

The Big Bang in T^3 Gowdy Cosmological Models

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Abstract

We establish a formal relationship between stationary axisymmetric spacetimes and T^3 Gowdy cosmological models which allows us to derive several preliminary results about the generation of exact cosmological solutions and their possible behavior near the initial singularity. In particular, we argue that it is possible to generate a Gowdy model from its values at the singularity and that this could be used to construct cosmological solutions with any desired spatial behavior at the Big Bang.

1 Introduction

The Hawking-Penrose theorems [1] prove that singularities are a generic characteristic of Einstein's equations. These theorems establish an equivalence between geodesic incompleteness and the blow up of some curvature scalars (the singularity) and allow us to determine the region (or regions) where singularities may exist. Nevertheless, these theorems say nothing about the nature of the singularities. This question is of great interest especially in the context of cosmological models, where the initial singularity (the “Big Bang”) characterizes the “beginning” of the evolution of the universe.

The first attempt to understand the nature of the Big Bang was made by Belinsky, Khalatnikov and Lifshitz [2]. They argued that the generic Big Bang is characterized by the mixmaster dynamics of spatially homogeneous Bianchi cosmologies of type VIII and IX. However, there exist counterexamples suggesting that the mixmaster behavior is no longer valid in models with more than three dynamical degrees of freedom [3]. Several alternative

behaviors have been suggested [4], but during the last two decades investigations have concentrated on the so-called asymptotically velocity term dominated (AVTD) behavior according to which, near the singularity, each point in space is characterized by a different spatially homogeneous cosmology [5]. The AVTD behavior of a given cosmological metric is obtained by solving the set of “truncated” Einstein’s equations which are the result of neglecting all terms containing spatial derivatives and considering only the terms with time derivatives.

Spatially compact inhomogeneous spacetimes admitting two commuting spatial Killing vector fields are known as Gowdy cosmological models [6]. Recently, a great deal of attention has been paid to these solutions as favorable models for the study of the asymptotic behavior towards the initial cosmological singularity. Since Gowdy spacetimes provide the simplest inhomogeneous cosmologies, it seems natural to use them to analyze the correctness of the AVTD behavior. In particular, it has been proved that all polarized Gowdy models belong to the class of AVTD solutions and it has been conjectured that the general (unpolarized) models are AVTD too [7].

In this work, we focus on T^3 Gowdy cosmological models and present the Ernst representation of the corresponding field equations. We show that this representation can be used to explore different types of solution generating techniques which have been applied very intensively to generate stationary axisymmetric solutions. We use this analogy to apply several known theorems to the case of Gowdy cosmological solutions. In particular, we use these results to show that all polarized T^3 Gowdy models preserve the AVTD behavior at the initial singularity and that “almost” all unpolarized T^3 Gowdy models can be generated from a given polarized seed solution by applying the solution generating techniques. We also analyze the possibility of generating a polarized model if we specify *a priori* any desired value of its Ernst potential at the initial singularity. We also argue that this method could be used to generate a cosmological model starting from its value at the Big Bang.

2 Gowdy T^3 Cosmological Models

Gowdy cosmological models are characterized by the existence of two commuting spatial Killing vector fields, say, $\eta_I = \partial/\partial\sigma$ and $\eta_{II} = \partial/\partial\delta$ which define a two parameter spacelike isometry group. Here σ and δ are spatial

coordinates delimited by $0 \leq \sigma, \delta \leq 2\pi$ as a consequence of the space topology. In the case of a T^3 -topology, the line element for unpolarized Gowdy models [6] can be written as

$$ds^2 = e^{-\lambda/2} e^{\tau/2} (-e^{-2\tau} d\tau^2 + d\theta^2) + e^{-\tau} [e^P (d\sigma + Q d\delta)^2 + e^{-P} d\delta^2] , \quad (1)$$

where the functions λ , P and Q depend on the coordinates τ and θ only, with $\tau \geq 0$ and $0 \leq \theta \leq 2\pi$. In the special case $Q = 0$, the Killing vector fields η_I and η_{II} become hypersurface orthogonal to each other and the metric (1) describes the polarized T^3 Gowdy models.

The corresponding Einstein's vacuum field equations consist of a set of two second order differential equations for P and Q

$$P_{\tau\tau} - e^{-2\tau} P_{\theta\theta} - e^{2P} (Q_\tau^2 - e^{-2\tau} Q_\theta^2) = 0 , \quad (2)$$

$$Q_{\tau\tau} - e^{-2\tau} Q_{\theta\theta} + 2(P_\tau Q_\tau - e^{-2\tau} P_\theta Q_\theta) = 0 , \quad (3)$$

and two first order differential equations for λ

$$\lambda_\tau = P_\tau^2 + e^{-2\tau} P_\theta^2 + e^{2P} (Q_\tau^2 + e^{-2\tau} Q_\theta^2) , \quad (4)$$

$$\lambda_\theta = 2(P_\theta P_\tau + e^{2P} Q_\theta Q_\tau) . \quad (5)$$

The set of equations for λ can be solved by quadratures once P and Q are known, because the integrability condition $\lambda_{\tau\theta} = \lambda_{\theta\tau}$ turns out to be equivalent to Eqs.(2) and (3).

To apply the solution generating techniques to the T^3 Gowdy models it is useful to introduce the Ernst representation of the field equations. To this end, let us introduce a new coordinate $t = e^{-\tau}$ and a new function $R = R(t, \theta)$ by means of the equations $R_t = t e^{2P} Q_\theta$, $R_\theta = t e^{2P} Q_t$. Then, the field equations (2) and (3) can be expressed as

$$t^2 \left(P_{tt} + \frac{1}{t} P_t - P_{\theta\theta} \right) + e^{-2P} (R_t^2 - R_\theta^2) = 0 , \quad (6)$$

$$t e^P \left(R_{tt} + \frac{1}{t} R_t - R_{\theta\theta} \right) - 2[(t e^P)_t R_t - (t e^P)_\theta R_\theta] = 0 . \quad (7)$$

Furthermore, this last equation for R turns out to be identically satisfied if the integrability condition $R_{t\theta} = R_{\theta t}$ is fulfilled. We can now introduce the complex Ernst potential ϵ and the complex gradient operator D as

$$\epsilon = te^P + iR, \quad \text{and} \quad D = \left(\frac{\partial}{\partial t}, i \frac{\partial}{\partial \theta} \right), \quad (8)$$

which allow us to write the main field equations in the *Ernst-like representation*

$$Re(\epsilon) \left(D^2 \epsilon + \frac{1}{t} Dt D\epsilon \right) - (D\epsilon)^2 = 0. \quad (9)$$

It is easy to see that the field equations (6) and (7) can be obtained as the real and imaginary part of the Ernst equation (9), respectively.

The particular importance of the Ernst representation (9) is that it is very appropriate to investigate the symmetries of the field equations. In particular, the symmetries of the Ernst equation for stationary axisymmetric spacetimes have been used to develop the modern solution generating techniques [12]. Similar studies can be carried out for any spacetime possessing two commuting Killing vector fields. Consequently, it is possible to apply the known techniques (with some small changes) to generate new solutions for Gowdy cosmological models. This task will be treated in a forthcoming work. Here, we will use the analogies which exist in spacetimes with two commuting Killing vector fields in order to establish some general properties of Gowdy cosmological models.

An interesting feature of Gowdy models is its behavior at the initial singularity which in the coordinates used here corresponds to the limiting case $\tau \rightarrow \infty$. The asymptotically velocity term dominated (AVTD) behavior has been conjectured as a characteristic of spatially inhomogeneous Gowdy models. This behavior implies that, at the singularity, all spatial derivatives in the field equations can be neglected in favor of the time derivatives. For the case under consideration, it can be shown that the AVTD solution can be written as [8]

$$P = \ln[\alpha(e^{-\beta\tau} + \zeta^2 e^{\beta\tau})], \quad Q = \frac{\zeta}{\alpha(e^{-2\beta\tau} + \zeta^2)} + \xi, \quad (10)$$

where α , β , ζ and ξ are arbitrary functions of θ . At the singularity, $\tau \rightarrow \infty$, the AVTD solution behaves as $P \rightarrow \beta\tau$ and $Q \rightarrow Q_0 = 1/(\alpha\zeta) + \xi$. It

has been shown [9] that that all polarized ($Q = 0$) T^3 Gowdy models have the AVTD behavior, while for unpolarized ($Q \neq 0$) models this has been conjectured. We will see in the following section that these results can be confirmed by using the analogy with stationary axisymmetric spacetimes.

3 Analogies and general results

Consider the line element for stationary axisymmetric spacetimes in the Lewis-Papapetrou form

$$ds^2 = -e^{2\psi}(dt + \omega d\phi)^2 + e^{-2\psi}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2] , \quad (11)$$

where ψ , ω , and γ are functions of the nonignorable coordinates ρ and z . The ignorable coordinates t and ϕ are associated with two Killing vector fields $\eta_I = \partial/\partial t$ and $\eta_{II} = \partial/\partial \phi$. The field equations take the form

$$\psi_{\rho\rho} + \frac{1}{\rho}\psi_\rho + \psi_{zz} + \frac{e^{4\psi}}{2\rho^2}(\omega_\rho^2 + \omega_z^2) = 0 , \quad (12)$$

$$\omega_{\rho\rho} - \frac{1}{\rho}\omega_\rho + \omega_{zz} + 4(\omega_\rho\psi_\rho + \omega_z\psi_z) = 0 , \quad (13)$$

$$\gamma_\rho = \rho(\psi_\rho^2 - \psi_z^2) - \frac{e^{4\psi}}{4\rho^2}(\omega_\rho^2 - \omega_z^2) , \quad (14)$$

$$\gamma_z = 2\rho\psi_\rho\psi_z - \frac{1}{2\rho}e^{4\psi}\omega_\rho\omega_z . \quad (15)$$

Consider now the following coordinate transformation $(\rho, t) \rightarrow (\tau, \sigma)$ and the complex change of coordinates $(\phi, z) \rightarrow (\delta, \theta)$ defined by

$$\rho = e^{-\tau}, \quad t = \sigma, \quad z = i\theta, \quad \phi = i\delta, \quad (16)$$

and introduce the functions P , Q and λ by means of the relationships

$$\psi = \frac{1}{2}(P - \tau), \quad Q = i\omega, \quad \gamma = \frac{1}{2}\left(P - \frac{\lambda}{2} - \frac{\tau}{2}\right) . \quad (17)$$

Introducing Eqs.(16) and (17) into the line element (11), we obtain the Gowdy line element (1), up to an overall minus sign. Notice that this method

for obtaining the Gowdy line element from the stationary axisymmetric one involves real as well as complex transformations at the level of coordinates and metric functions. It is, therefore, necessary to demand that the resulting metric functions be real. Indeed, one can verify that the action of the transformations (16) and (17) on the field equations (12)-(15) yields exactly the field equations (2)-(5) for the Gowdy cosmological models. This is an interesting property that allows us to generalize several results known for stationary axisymmetric spacetimes to the case of Gowdy spacetimes.

The counterparts of static axisymmetric solutions ($\omega = 0$) are the polarized ($Q = 0$) Gowdy models. For instance, the Kantowski-Sachs [10] cosmological model is the counterpart of the Schwarzschild spacetime, one of the simplest static solutions. Furthermore, it is well known that the field equations for static axisymmetric spacetimes are linear and there exists a general solution which can be generated (by using properties of harmonic functions) from the Schwarzschild one [11]. According to the analogy described above, this implies the following

Lemma 1: All polarized T^3 Gowdy cosmological models can be generated from the Kantowski-Sachs solution.

The method for generating polarized Gowdy models can be briefly explained in the following way. If we introduce the time coordinate $t = e^{-\tau}$, Eq. (2) becomes $\Delta P = 0$ with $\Delta = \partial_{tt} + t^{-1}\partial_t - \partial_{\theta\theta}$. It can easily be verified that the operators $L_1 = t^{-1}\partial_t(t\partial_t)$ and $L_2 = \partial_\theta$ commute with the operator Δ . Then if a solution P_0 is known ($\Delta P_0 = 0$), the action of the operator L_i ($i = 1, 2$) on P_0 generates new solutions \tilde{P}_i , i.e., if $\tilde{P}_i = L_i P_0$ then $\Delta \tilde{P}_i = \Delta L_i P_0 = 0$. This procedure can be repeated as many times as desired, generating an infinite number of solutions whose sum with arbitrary constant coefficients represents the general solution. An important property of the action of L_i on a given solution P_0 is that it preserves the behavior of P_0 for $\tau \rightarrow \infty$. If we choose the Kantowski-Sachs spacetime as the seed solution P_0 , the general solution will be AVTD. Hence, as a consequence of Lemma 1, we obtain

Lemma 2: All polarized T^3 Gowdy cosmological models are AVTD.

We now turn to the general unpolarized ($Q \neq 0$) case. The solution generating techniques have been applied intensively to generate stationary solutions from static ones. In particular, it has been shown that “almost” all stationary axisymmetric solutions can be generated from a given static solution (the Schwarzschild spacetime, for instance) [12]. The term “almost”

means that there exist “critical” points where the field equations are not well defined and, therefore, the solution generating techniques can not be applied. Using the analogy with Gowdy models, we obtain

Lemma 3: “Almost” all unpolarized T^3 Gowdy cosmological solutions can be generated from a given polarized seed solution.

As in the previous case, it can be shown that the solution generating techniques preserve the asymptotic behavior of the seed solution. If we take the Kantowski-Sachs spacetime as seed solution and apply Lemma 3, we obtain

Lemma 4: “Almost” all T^3 unpolarized Gowdy models can be generated from the Kantowski-Sachs solution and are AVTD.

It should be mentioned that all the results presented in Lemma 1 - 4 must be treated as “preliminary”. To “prove” them we have used only the analogy between stationary axisymmetric spacetimes and Gowdy cosmological models, based on the transformations (16) and (17). A rigorous proof requires a more detailed investigation and analysis of the symmetries of the equations (2)-(5), especially in their Ernst representation (9).

4 The Big Bang

The behavior of the Gowdy T^3 cosmological models at the initial singularity ($\tau \rightarrow \infty$) is dictated by the AVTD solution (10). Although this is not the only possible case for the Big Bang, there are physical reasons to believe that this is a suitable scenario for the simplest case of inhomogeneous cosmological models. But an inhomogeneous Big Bang has two different aspects. The first one concerns the temporal behavior as the singularity is approached. The second aspect is related to the spatial inhomogeneities which should be present during the Big Bang. If we accept the AVTD behavior, the first aspect of the problem becomes solved by the AVTD solution (10), which determines the time dependence at the Big Bang. However, the spatial dependence remains undetermined as it is given by the arbitrary functions α , β , ζ and ξ which can be specified only once a solution is known. The question arises whether it is possible to have a solution with any desired spatial behavior at the Big Bang. We will see that the answer to this question is affirmative.

Using the solution generating techniques for stationary axisymmetric spacetimes it has been shown that any solution can be generated from its

values on the axis of symmetry ($\rho = 0$) [12]. Specific procedures have been developed that allow us to construct any solution once the value of the corresponding Ernst potential is given at the axis [13]. On the other hand, the analogy with Gowdy models determined by Eqs.(16) and (17) indicates that the limit $\rho \rightarrow 0$ is equivalent to $\tau \rightarrow \infty$. To see that this equivalence is also valid for specific solutions we write the counterpart of the AVTD solution (10) in the coordinates ρ and z , according to the Eqs.(16) and (17). Then, we obtain

$$\psi = \frac{1}{2} \ln[\alpha(\rho^{1+\beta} + \zeta^2 \rho^{1-\beta})], \quad \omega = \frac{\zeta}{\alpha(\rho^{2\beta} + \zeta^2)} + \xi, \quad (18)$$

where we have chosen the arbitrary functions ζ and ξ such that ω becomes a real function. One can verify that the solution (18) satisfies the corresponding “AVTD equations” (12) and (13) (dropping the derivatives with respect to z) for the stationary case. Consequently, the behavior of stationary axisymmetric solutions at the axis ($\rho \rightarrow 0$) is equivalent to the behavior at T^3 Gowdy models at the Big Bang.

Thus, if we consider the asymptotic behavior of the Ernst potential (9) for the AVTD solution (10)

$$\epsilon(\tau \rightarrow \infty) \rightarrow \epsilon_0 = e^{\tau(\beta-1)}[1 + Q_0 e^{\tau(\beta-1)}], \quad Q_0 = \frac{1}{\alpha\zeta} + \xi, \quad (19)$$

and specify α , β , ζ and ξ as functions of the spatial coordinate θ , it is possible to derive the corresponding unpolarized ($Q \neq 0$) solution by using the solution generating techniques. In other words, we can construct a Gowdy model with any desired behavior at the Big Bang. This is an interesting possibility which could be used to analyze physically reasonable scenarios for the Big Bang.

Of course, a more detailed study of the solution generating techniques for Gowdy cosmological models is necessary in order to provide a rigorous proof of these results and to construct realistic models for the Big Bang. Here we have used the analogy between stationary axisymmetric spacetimes and T^3 Gowdy cosmological models to show that this is a task that in principle can be solved.

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